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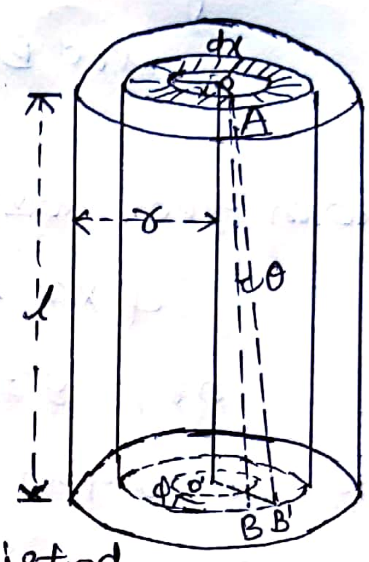
B.Sc (Part-1) Hons

Twisting Couple on a Cylindrical Rod



Let a cylindrical rod of length l and radius r be hung vertically with its upper end rigidly fixed. Let O' and O be the respective centres of its upper and lower ends. Let the rod be twisted by applying an external couple of moment τ at its lower end in the horizontal plane. As the rod is twisted an elastic restoring couple begins to develop in it. When this couple equals the twisting couple, the rod comes in the equilibrium state. Let ϕ radian be the angle of twist at the lower end of the rod in the equilibrium state.

Let us divide the cylindrical rod into a large number of thin co-axial cylindrical shells and consider one such elementry shell of radius x and thickness dx. The generating line AB on the surface of this shell has been rotated in to the position AB'. $\angle BAB'$ is the angle of shear θ for this shell. In the curved triangle OBB', we have

$$\text{arc } BB' = \phi \times OB \quad (\text{arc} = \text{angle} \times \text{radius})$$

$$= \phi \times x$$

Similarly the curved triangle ABB'; we have

$$\text{arc } BB' = \theta \times AB$$

$$= \theta \times l$$

from the last two expression, we have

$$\phi \times x = \theta \times l$$

$$\text{or,} \quad \theta = \frac{x\phi}{l} \quad \text{——} (1)$$

Let F be the tangential force acting over the base of the elementry shell. It is circular annulus of area

$$\pi(x+dx)^2 - \pi x^2 = 2\pi x\, dx$$

neglecting the smaller term containing $(dx)^2$. Hence

$$\text{tangential stress} = \frac{\text{force}}{\text{area}}$$

$$= \frac{F}{2\pi x\, dx} \quad \text{——} (2)$$

Then the modulus of rigidity 'η' of the material of the rod is

$$\eta = \frac{\text{tangential stress}}{\text{Shear } \theta}$$

$$\eta = \frac{F}{2\pi x\, dx} \cdot \frac{l}{x\phi} \quad (\text{from eqn (1) and (2)})$$

$$\text{or,} \quad F = \frac{2\pi\eta\phi}{l} x^2 dx$$

The momental of this force about the axis OO' of the rod is

$$d\tau = F \cdot x$$

$$= \frac{2\pi\eta\phi}{l} x^3 dx$$

This is equal to the Couple necessary to twist the elementry shell through angle ϕ.

The Couple, τ necessary to twist the total rod will be obtained by integrating the last expression the limit $x = 0$ to $x = r$

$$\therefore \tau = \int_0^r \frac{2\pi\eta\phi}{l} \cdot x^3 dx$$

$$= \frac{2\pi\eta\phi}{l} \int_0^r x^3 dx$$

$$= \frac{2\pi\eta\phi}{l} \left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{2\pi\eta\phi}{l} \cdot \frac{r^4}{4}$$

$$\therefore \tau = \frac{\pi\eta r^4}{2l} \phi \quad \text{——— (3)}$$

This shows that the Couple necessary to twist the rod through ϕ radian is proportional to the twist ϕ.

The Couple necessary to Cause a twist of one radian is, therefore given by

$$C = \frac{\pi\eta r^4}{2l} \quad \text{——— (4)}$$

c is called the 'torsional rigidity' or 'torsional constant' of the wire. It is also called as the restoring couple per unit twist, since twisting couple is numerically equal to the restoring couple.

The couple necessary to twist the rod through 90° (i.e. $\pi/2$ radian) will be from equation (3)

$$\frac{\pi \eta r^4}{2l}\left(\frac{\pi}{2}\right) = \frac{\pi^2 \eta r^4}{4l} \quad \text{— (5)}$$

From equation (1) the shearing strain θ of the outermost layer of the wire of the length l and radius r is

$$\theta = \frac{r\phi}{l}$$ (where ϕ is the angle of twist)

$$\therefore \boxed{\phi = \frac{l\theta}{r}}$$